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ABSTRACT. According to Chen and Fu's method, we offer a short proof for Dougall's ${}_2H_2$ -series identity.

Recently, Chen and Fu [3] gave the semi-finite forms of several q -series identities in a surprising method. Inspired by this work, we shall derive directly Dougall's ${}_2H_2$ -series identity form a nonterminating form of Saalschütz's theorem in the same method.

For a complex number x and an integer n , define the shifted factorial by

$$(x)_n = \Gamma(x+n)/\Gamma(x)$$

where Γ -function is well-defined

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \text{ with } \Re(x) > 0.$$

For simplifying the expressions, we introduce the following fraction forms of them:

$$\Gamma \left[\begin{matrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{matrix} \right] = \frac{\Gamma(a)\Gamma(b)\cdots\Gamma(c)}{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)},$$
$$\left[\begin{matrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{matrix} \right]_n = \frac{(a)_n(b)_n\cdots(c)_n}{(\alpha)_n(\beta)_n\cdots(\gamma)_n}.$$

Then a bilateral summation formula due to Dougall [5](see also Slater [7, p. 181]) can be stated as follows.

Theorem 1 (Dougall's ${}_2H_2$ -series identity).

$${}_2H_2 \left[\begin{matrix} a, & b \\ c, & d \end{matrix} \middle| 1 \right] = \sum_{k=-\infty}^{\infty} \left[\begin{matrix} a, & b \\ c, & d \end{matrix} \right]_k = \Gamma \left[\begin{matrix} 1-a, & 1-b, & c, & d, & c+d-a-b-1 \\ c-a, & c-b, & d-a, & d-b \end{matrix} \right]$$

where $\Re(c+d-a-b) > 1$.

The original proof for Theorem 1 given by Dougall [5] depends on Cauchy residue theorem. For other three different proofs, the reader may refer to Andrews et al. [1, p. 110], Chu [4] and Slater [7, p. 181] respectively. Now, a short new proof for Theorem 1 will subsequently be offered.

Proof. Following Bailey [2], define the unilateral hypergeometric series by

$${}_{1+r}F_s \left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_s \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ 1, & b_1, & \cdots, & b_s \end{matrix} \right]_k z^k.$$

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Then a nonterminating form of Saalschütz's theorem(cf. Andrews et al. [1, p. 92]) can be expressed as

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} c+d-a-b-1, a, b \\ c, d \end{matrix} \middle| 1 \right] &= {}_3F_2 \left[\begin{matrix} c-a, c-b, 1 \\ c-a-b+1, c+d-a-b \end{matrix} \middle| 1 \right] \\ &\times \Gamma \left[\begin{matrix} c, d \\ a, b, c+d-a-b \end{matrix} \right] \frac{1}{a+b-c} + \Gamma \left[\begin{matrix} c, d, c-a-b, d-a-b \\ c-a, c-b, d-a, d-b \end{matrix} \right] \end{aligned}$$

where $\Re(d-a-b) > 0$. Performing the replacements $k \rightarrow k+n$, $a \rightarrow a-n$, $b \rightarrow b-n$, $c \rightarrow c-n$ and $d \rightarrow d-n$ for the last equation where k denotes the summation index of the ${}_3F_2$ -series on the left hand side, we have

$$\begin{aligned} \sum_{k=-n}^{\infty} \left[\begin{matrix} c+d-a-b-1, a-n, b-n \\ 1, c-n, d-n \end{matrix} \right]_{k+n} &= {}_3F_2 \left[\begin{matrix} c-a, c-b, 1 \\ c-a-b+1+n, c+d-a-b \end{matrix} \middle| 1 \right] \\ &\times \Gamma \left[\begin{matrix} c-n, d-n \\ a-n, b-n, c+d-a-b \end{matrix} \right] \frac{1}{a+b-c-n} + \Gamma \left[\begin{matrix} c-n, d-n, c-a-b+n, d-a-b+n \\ c-a, c-b, d-a, d-b \end{matrix} \right] \end{aligned}$$

which is equivalent to the identity

$$\begin{aligned} \sum_{k=-n}^{\infty} \left[\begin{matrix} c+d-a-b-1+n, a, b \\ 1+n, c, d \end{matrix} \right]_k &= {}_3F_2 \left[\begin{matrix} c-a, c-b, 1 \\ c-a-b+1+n, c+d-a-b \end{matrix} \middle| 1 \right] \\ &\times \Gamma \left[\begin{matrix} 1+n \\ c+d-a-b+n \end{matrix} \right] \Gamma \left[\begin{matrix} c, d \\ a, b \end{matrix} \right] \frac{c+d-a-b-1+n}{(a+b-c-n)(c+d-a-b-1)} \\ &+ \Gamma \left[\begin{matrix} 1+n, c-a-b+n, d-a-b+n \\ 1-a+n, 1-b+n, c+d-a-b-1+n \end{matrix} \right] \Gamma \left[\begin{matrix} 1-a, 1-b, c, d, c+d-a-b-1 \\ c-a, c-b, d-a, d-b \end{matrix} \right]. \end{aligned} \quad (1)$$

Recall the limiting relation on Γ -function:

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n+x)}{\Gamma(n+y)} n^{y-x} = 1.$$

Letting $n \rightarrow \infty$ for (1) and noting that the first term on the right hand side vanishes under the condition $\Re(c+d-a-b) > 1$, we get directly Theorem 1 to complete the proof. \square

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